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Potential Flow of a Film Down an Inclined Plate

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SUMMARY
The flow of a liquid film along a semi-infinite flat plate due to gravity is considered, the fluid being assumed inviscid and incompressible. When the Froude number *Fr*, based on the initial film thickness and velocity, is large compared to unity, solutions can be found by the method of matched asymptotic expansions. The fluid speed and deflection, and the pressure gradient are found to $O(Fr^{-2})$. Hydraulic theory enters as the first term in the outer expansion, which is valid far downstream from the leading edge. When the liquid falls vertically, the motion represents half of a freely falling jet.

1. Introduction

The shooting motion of a liquid film along an inclined plate under the influence of gravity occurs often in nature and has many practical applications. Hydraulic theory provides a useful first approximation to this flow when viscous effects are negligible (e.g., when the extent of the motion is over a relatively short distance) and hydraulic jumps are absent. Keller and Weitz [1] sought to improve this theory by finding higher approximations, but a number of arbitrary constants arise which must be determined from initial conditions upstream. This paper considers a liquid film which is introduced at the leading edge of a semi-infinite flat plate with uniform supercritical speed q_0 and initial thickness b_0. We seek potential solutions which are uniformly valid for large values of the Froude number $Fr = q_0^2/gb_0 \equiv 1/\varepsilon$, g being the acceleration of gravity. The effects of surface tension are ignored because the streamline curvature nondimensionalized with respect to b_0, is of the $O(\varepsilon^2)$ throughout the film (see Sec. 5).

Recently Clarke [2] used the method of matched expansions to solve for the two-dimensional flow over a waterfall. The hydraulic approximation entered as the first term in an outer expansion, valid far downstream from the edge. The inner expansion satisfied a uniform stream condition at an infinite distance upstream and the unknown constants in the outer expansion were determined by the matching technique. In practical cases, the initial conditions are applied at a place which, in the scale of the experiment, does not appear to be infinitely remote. It is of interest to know how this finiteness modification affects the simple hydraulic theory and its higher approximations. Ackerberg [3] has found that the determination of the viscous boundary layer flow along the plate depends on the pressure gradient in the potential flow which exists before the boundary layer penetrates the free surface. This pressure gradient can only be known with certainty by working out higher approximations to the hydraulic theory and this provided the motivation for this study. However, when the liquid falls vertically, the potential flow also represents half of a free falling jet, and this can readily be observed in nature without a plate boundary layer. There will, however, be very weak boundary layers along the free surface in which the vorticity will be of $O(u_0/b_0 Fr^2)$. For $Fr \gg 1$, this is a small effect which can be neglected. Although the expansion methods in this paper are similar to some of those used by Ackerberg [4] in the study of jets, a number of new features arise here in applying initial conditions to problems in hydraulics.

The stability of this flow for a viscous fluid is an open question. All stability analyses are based on the asymptotic shear flow which would be obtained after a potential flow, like the one considered here, is absorbed by the boundary layer. Visual observations (by the author) indicate that a smooth flow does exist along the initial portion of a spillway surface and it is likely that these potential flows do occur in nature.

In Sec. 2 the problem is formulated mathematically in the plane of the complex velocity potential using the logarithm of the complex velocity as the dependent variable. The outer expansion, based on streamwise distances of the $O(b_0/\varepsilon)$, is found in Sec. 3 using the hydraulic approximation as the lowest order term. The initial condition cannot be satisfied by this expansion to $O(\varepsilon^2)$ for any choice of the arbitrary constants and an inner expansion is required to remove this non-uniformity. Matching of the expansions fixes two unknown constants which appear in the outer expansion to $O(\varepsilon^2)$. Finally in Sec. 5, the streamline curvature and the pressure gradient are determined to $O(\varepsilon^2)$ using composite expansions constructed from the inner and outer expansions.

2. Mathematical Formulation

Choose a coordinate system $Z=X+iY$ with the plate lying along the $+X$-axis and origin at the leading edge (see Fig. 1). Gravity acts in a direction which makes an angle α with the plate. A complex velocity potential $W(Z)=\Phi+i\Psi$ and a complex velocity $dW/dZ=U-iV$ must be

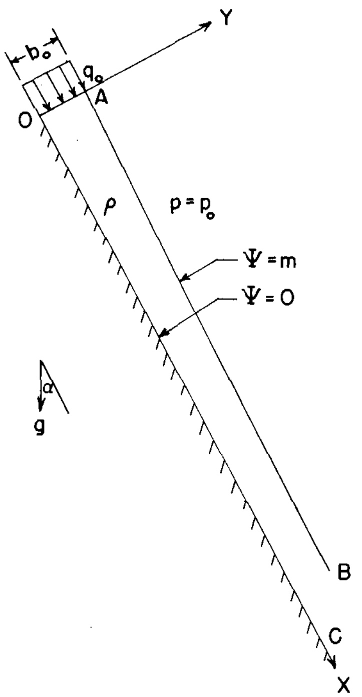

Figure 1. Flow Geometry.

found such that: 1) the pressure on the free surface AB is constant; 2) V vanishes along the plate $Y=0$; 3) at the leading edge $X=0$, the fluid speed $q=q_0$ for $0\leqq Y\leqq b_0$. The assumption $Fr\gg 1$ obviates the necessity of specifying boundary conditions downstream which are usually required in potential problems.

Non-dimensionalize the coordinates, the complex velocity potential, and the complex velocity as follows:

$$z=x+iy=Z/(m/q_0), \quad w=\varphi+i\psi=W/m, \quad dw/dz=u-iv=qe^{-i\theta}=q_0^{-1}\,dW/dZ\,. \tag{2.1}$$

Here m is the volumetric flux per unit breadth which equals b_0q_0 to first order, as we will see. Since the location of the free surface AB in the physical plane is unknown to start with, it is convenient to formulate this problem in the plane of the complex velocity potential. The image of the physical plane is shown in Fig. 2; by an appropriate choice of constants the origins can be made to correspond. The curved line OA which is the trace of $x=0$, $y \geqq 0$ will be discussed later.

Introduce the logarithm of the complex velocity

$$\Gamma(w;\varepsilon)=\ln(dw/dz)=Q(\varphi,\psi;\varepsilon)-i\theta(\varphi,\psi;\varepsilon)\,. \tag{2.2}$$

By the usual arguments the real and imaginary parts of Γ satisfy the Cauchy–Riemann equations and are conjugate harmonic functions of (φ, ψ).

Boundary Conditions

Along the wall OC the deflection is fixed; thus,

$$\theta(\varphi,0)=0 \quad \text{for} \quad \varphi \geqq 0\,. \tag{2.3}$$

The initial condition requires

$$Q=0 \quad \text{for} \quad x=0\,, \quad 0 \leqq y \leqq 1\,. \tag{2.4}$$

To apply this equation in the w-plane, we require the image of the line $x=0$, $0 \leqq y \leqq 1$. This may be found using (2.1) when

$$dx=\mathrm{Re}\,(e^{i\theta}\,dw/q)=0\,. \tag{2.5}$$

Thus, the integral of the equation

$$d\varphi/d\psi=\tan\theta(\varphi,\psi) \quad \text{with} \quad \varphi=0 \quad \text{when} \quad \psi=0\,, \tag{2.6}$$

fixes the line OA in the w-plane. Since (2.6) involves the unknown function θ, the integration cannot usually be carried out. In the asymptotic analysis given here, the lowest order term of θ

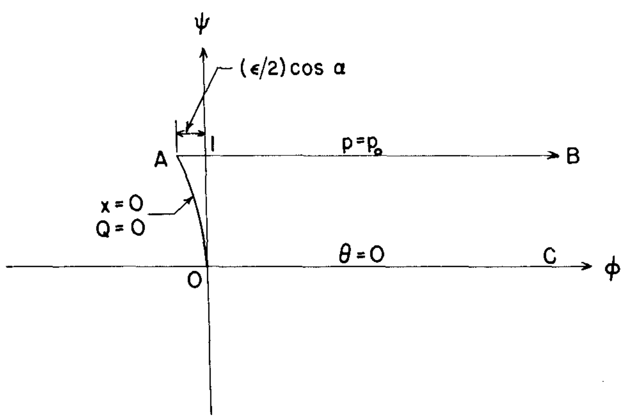

Figure 2. Plane of the Complex Velocity Potential w.

is particularly simple and the first approximation to this line is easily found.

On the free surface AB the pressure is constant. From Bernoulli's equation

$$p/\rho+\tfrac{1}{2}q_0^2q^2-b_0g(x\cos\alpha-y\sin\alpha)=\mathscr{H}\,, \tag{2.7}$$

where the total head $\mathscr{H}$ has the same value for all streamlines. This value may be determined from conditions at A $(x=0,\ y\approx 1)$ where $q=1$ and $p=p_0$; thus,

$$\mathscr{H}=p_0/\rho+\tfrac{1}{2}q_0^2+b_0g\sin\alpha\,. \tag{2.8}$$

A useful boundary condition in terms of Q and θ may be derived from (2.7) by differentiating partially with respect to φ, putting $\psi=1$, and noting $(\partial p/\partial\varphi)_{\psi=1}=0$; thus

$$q(\partial q/\partial\varphi)-\varepsilon[\cos\alpha(\partial x/\partial\varphi)-\sin\alpha(\partial y/\partial\varphi)]=0 \quad \text{on} \quad \psi=1\,. \tag{2.9}$$

Using $q=e^Q$ and (2.1), this may be written

$$\partial Q/\partial\varphi - \varepsilon e^{-3Q}(\cos\alpha\cos\theta - \sin\alpha\sin\theta) = 0 \quad \text{on} \quad \psi = 1\,. \tag{2.10}$$

This is Levi–Civita's form of the free surface condition. Therefore, the mathematical problem is to find harmonic functions Q and θ which satisfy the boundary conditions (2.3), (2.4), and (2.10). Since (2.10) is non-linear, this is too difficult for the most general case. When $\varepsilon \ll 1$ asymptotic solutions can be found by the method of matched expansions.

3. The hydraulic approximation and the outer expansion

According to hydraulic theory, the fluid will be in free fall in the first approximation. Therefore,

$$u = (1+2\varepsilon x\cos\alpha)^{\frac{1}{2}}\,, \qquad v = o(1)\,, \tag{3.1}$$

and from (2.7) the pressure variation to first order is hydrostatic. The initial condition (2.4) is satisfied to lowest order and this first approximation is uniformly valid. To improve this theory, it is convenient to use (φ, ψ) as independent variables. Noting that $\boldsymbol{q}=\nabla\varphi$, (3.1) asserts that φ does not vary in the y-direction to first order. Thus, $u=\partial\varphi/\partial x$ can be integrated to give $\varphi(x)$ [with $\varphi(0)=0$] and $q(\varphi)\approx u(\varphi)$ may be found. Carrying this out, we obtain to first order

$$Q(\varphi, \psi) = \ln q \approx (\tfrac{1}{3})\ln(1+3\varepsilon\varphi\cos\alpha)\,. \tag{3.2}$$

This solution suggests that a new independent variable should be introduced in place of φ which remains of $O(1)$ when φ is large; therefore, define

$$\varphi^* = \varepsilon\varphi\,. \tag{3.3}$$

ψ, being a measure of the volumetric flux in the film, remains of $O(1)$ and need not be scaled.

The outer expansion results formally by assuming

$$Q(\varphi, \psi; \varepsilon) \sim Q_0(\varphi^*, \psi) + \varepsilon Q_1(\varphi^*, \psi) + \varepsilon^2 Q_2(\varphi^*, \psi) + \dots\,, \tag{3.4}$$

$$\theta(\varphi, \psi; \varepsilon) \sim \varepsilon\theta_0(\varphi^*, \psi) + \varepsilon^2\theta_1(\varphi^*, \psi) + \varepsilon^3\theta_2(\varphi^*, \psi) + \dots\,. \tag{3.5}$$

Expressing the Cauchy–Riemann equations in the variables (φ^*, ψ) yields

$$\varepsilon\partial Q/\partial\varphi^* = -\partial\theta/\partial\psi\,, \tag{3.6}$$

$$\partial Q/\partial\psi = \varepsilon\partial\theta/\partial\varphi^*\,. \tag{3.7}$$

Substituting (3.4) and (3.5) into (3.6) and (3.7) and equating the coefficients of each power of ε to zero gives

$$\partial Q_0/\partial\psi = 0\,, \tag{3.8}$$

$$\partial Q_1/\partial\psi = 0\,, \tag{3.9}$$

$$\partial\theta_0/\partial\psi = -\partial Q_0/\partial\varphi^*\,, \tag{3.10}$$

$$\partial\theta_1/\partial\psi = -\partial Q_1/\partial\varphi^*\,, \tag{3.11}$$

$$\partial Q_2/\partial\psi = \partial\theta_0/\partial\varphi^*\,. \tag{3.12}$$

After integration, these equations yield

$$Q_0 = Q_0(\varphi^*)\,, \tag{3.13}$$

$$Q_1 = Q_1(\varphi^*)\,, \tag{3.14}$$

$$\theta_0 = -\psi Q_0'(\varphi^*)\,, \tag{3.15}$$

$$\theta_1 = -\psi Q_1'(\varphi^*)\,, \tag{3.16}$$

$$Q_2 = -(\tfrac{1}{2})\psi^2 Q_0''(\varphi^*) + a(\varphi^*)\,. \tag{3.17}$$

Here primes denote differentiation with respect to φ^* and $a(\varphi^*)$ is an unknown function to be determined. In obtaining (3.15) and (3.16) arbitrary functions of φ^* were omitted to satisfy the boundary condition (2.3).

When (2.10) is transformed to the variables (φ^*, ψ), we obtain after substituting the expansions (3.4) and (3.5), using (3.13)–(3.17), and equating the coefficients of each power of ε to zero

$$Q_0' = e^{-3Q_0} \cos \alpha \,, \tag{3.18}$$

$$Q_1' + 3e^{-3Q_0} Q_1 \cos \alpha = e^{-3Q_0} Q_0' \sin \alpha \,, \tag{3.19}$$

$$a' + 3e^{-3Q_0} a \cos \alpha = \tfrac{1}{2}\{Q_0''' + e^{-3Q_0}[3Q_0'' \cos \alpha - Q_0'^2 \cos \alpha + 2Q_1' \sin \alpha + 9\,Q_1^2 \cos \alpha - 6\,Q_0' Q_1 \sin \alpha]\} \,. \tag{3.20}$$

After integration we obtain

$$Q_0 = (\tfrac{1}{3}) \ln p \,, \tag{3.21}$$

$$Q_1 = (\sin \alpha/3\,p) \ln p + c_1/p \,, \tag{3.22}$$

$$a = -(6\,p^2)^{-1}\{(8 \cos^2 \alpha + 2 \sin^2 \alpha - 12\,c_1 \sin \alpha + 9\,c_1^2) + (6\,c_1 \sin \alpha - 4 \sin^2 \alpha)(1 + \ln p) + \sin^2 \alpha[(1 + \ln p)^2 + 1]\} + c_2/p \,, \tag{3.23}$$

where

$$p(\varphi^*) = c + 3\,\varphi^* \cos \alpha \,, \tag{3.24}$$

and c, c_1, and c_2 are constants of integration. A comparison of (3.21) with (3.2) shows that $c=1$.

To apply the initial condition (2.4), a solution of (2.6) is required. Substituting (3.5), (3.15), and (3.21) into (2.6) we obtain after integrating, applying the boundary condition, and retaining the highest order term

$$\varphi^* \approx -\varepsilon^2 (\cos \alpha/2)\psi^2 \quad \text{on} \quad x = 0,\ y \geqq 0 \,. \tag{3.25}$$

Substituting this result into (3.4), using (3.21)–(3.24) and expanding for $\varepsilon \to 0$, we obtain

$$Q \sim \varepsilon c_1 + \varepsilon^2[\psi^2 \cos^2 \alpha + a\,(p=1)] + O(\varepsilon^3) \quad \text{on} \quad x = 0,\ y \geqq 0 \,, \tag{3.26}$$

where

$$a\,(p=1) = -(8 \cos^2 \alpha - 6\,c_1 \sin \alpha + 9\,c_1^2)/6 + c_2 \,. \tag{3.27}$$

The boundary condition (2.4) requires that the coefficients of each power of ε in (3.26) vanish for $0 \leqq \psi \leqq 1$. For $\alpha \neq \pi/2$, this is clearly impossible and our method of successive approximation has failed. At best we could choose $c_1 = 0$ and satisfy the boundary condition to $O(\varepsilon)$; however, this value of c_1 will be obtained from the matching procedure.

4. The inner solution

To satisfy the initial condition (2.4) an inner solution, valid near the leading edge, is required. The characteristic length in this region will be b_0, and the asymptotic expansions should be based on the independent variables (φ, ψ). The first term in the inner expansion for Q can be deduced from $Q_0(\varphi^*)$ when $\varphi^* \to 0$ [see (3.21)] because of its uniform validity. Therefore, we assume

$$Q(\varphi, \psi; \varepsilon) \sim \varepsilon\varphi \cos \alpha + \varepsilon^2 Q^+(\varphi, \psi) + \ldots \tag{4.1}$$

$$\theta(\varphi, \psi; \varepsilon) \sim -\varepsilon\psi \cos \alpha + \varepsilon^2 \theta^+(\varphi, \psi) + \ldots \tag{4.2}$$

From (2.2) it follows that Q^+ and θ^+ are harmonic; thus

$$\nabla^2 Q^+ = 0 = \nabla^2 \theta^+ \,. \tag{4.3}$$

The boundary condition (2.3) requires

$$\theta^+(\varphi, 0) = 0 \quad \text{for} \quad \varphi > 0 .$$

or in terms of Q^+

$$\partial Q^+/\partial\psi = 0 \quad \text{for} \quad \varphi > 0, \ \psi = 0 . \tag{4.4}$$

Substituting (4.1) and (4.2) into (2.10) we obtain

$$\partial Q^+/\partial\varphi = \sin\alpha\cos\alpha - 3\varphi\cos^2\alpha \quad \text{on} \quad \psi = 1 . \tag{4.5}$$

If (4.2) is substituted into (2.6) and integrated, we obtain (3.25) with φ^* replaced by $\varepsilon\varphi$, i.e.,

$$\varphi = -\varepsilon(\cos\alpha/2)\psi^2 \quad \text{on} \quad x = 0, \ y \geqq 0 . \tag{4.6}$$

This is the line OA shown in Fig. 2. The initial condition (2.4) will be satisfied if

$$Q(\varphi, \psi; \varepsilon) = 0 \quad \text{when} \quad \varphi = -\varepsilon(\cos\alpha/2)\psi^2, \qquad 0 \leqq \psi \leqq 1 , \tag{4.7}$$

for each power of ε. When (4.1) and (4.6) are substituted into (4.7) we obtain

$$\varepsilon^2\{Q^+[O(\varepsilon), \psi] - \psi^2(\cos^2\alpha/2)\} + o(\varepsilon^2) = 0 . \tag{4.8}$$

If Q^+ is regular when $\varphi \to 0$, this becomes

$$Q^+(0, \psi) = \psi^2(\cos^2\alpha/2) \quad \text{for} \quad 0 \leqq \psi \leqq 1 . \tag{4.9}$$

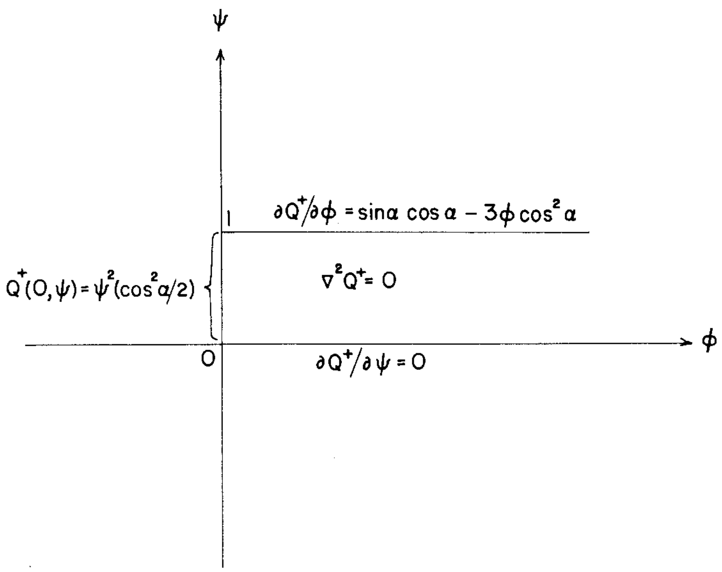

Figure 3. Boundary Value Problem for Q^+.

A harmonic function Q^+ which satisfies the boundary conditions (4.4), (4.5), and (4.9), and which diverges least rapidly when $\varphi \to \infty$ (see Fig. 3) is

$$Q^+(\varphi, \psi) = (3\cos^2\alpha/2)(\psi^2 - \varphi^2) + \varphi\sin\alpha\cos\alpha - \cos^2\alpha + H(\varphi, \psi)\cos^2\alpha , \tag{4.10}$$

where H is a solution of the following boundary value problem:

$$\nabla^2 H = 0 \qquad 0 \leqq \varphi \leqq \infty , \qquad 0 \leqq \psi \leqq 1 , \tag{4.11}$$

$$H(\varphi, 1) = 0 \qquad \varphi \geqq 0 , \tag{4.12}$$

$$\partial H/\partial\psi = 0 \quad \text{on} \quad \psi = 0, \quad \varphi \geqq 0 , \tag{4.13}$$

$$H(0,\psi)=1-\psi^2 \qquad 0\leqq\psi\leqq 1\,, \tag{4.14}$$

$$H(\varphi,\psi)\to 0 \quad \text{for} \quad \varphi\to\infty, \qquad 0\leqq\psi\leqq 1\,. \tag{4.15}$$

On solving for H by separation of variables we obtain

$$H(\varphi,\psi)=\sum_{n=0}^{\infty} d_n \exp[-(2n+1)\pi\varphi/2]\cos[(2n+1)\pi\psi/2]\,, \tag{4.16}$$

where

$$d_n=4(-1)^n\left(\frac{2n+1}{2}\pi\right)^{-3}\,. \tag{4.17}$$

This series converges absolutely and it follows that

$$|H(\varphi,\psi)|\leqq M\exp(-\pi\varphi/2) \quad \text{when} \quad 0\leqq\psi\leqq 1\,, \tag{4.18}$$

for some constant M. Since the matching procedure is carried out for $\varphi\to\infty$ in the inner expansion, H contributes only exponentially small terms which can be neglected.

The complex conjugate function θ^+ can be found from (4.10) and (4.16) using the Cauchy-Riemann equations; thus

$$\theta^+(\varphi,\psi)=(3\cos^2\alpha)\varphi\psi-\psi\sin\alpha\cos\alpha$$

$$+\cos^2\alpha\sum_{n=0}^{\infty} d_n\exp[-(2n+1)\pi\varphi/2]\sin[(2n+1)\pi\psi/2]\,. \tag{4.19}$$

The upper bound (4.18) applies to this series as well.

The matching procedure

Although the step-by-step matching procedure proposed by Van Dyke [5] could be applied, the following equivalent method is used here: i) Let $\varphi\to\infty$ in the inner expansion of Q and, neglecting exponentially small terms, express what remains in terms of φ^*; ii) expand the outer expansion of Q for $\varphi^*\to 0$; iii) the arbitrary constants appearing in (ii) must be chosen so that for every term in (i) a corresponding term appears in (ii).

Carrying out (i) we obtain

$$\begin{aligned}Q_{\text{inner}}&\to\varphi^*\cos\alpha-(3\cos^2\alpha/2)\varphi^{*2}+\varepsilon\varphi^*\sin\alpha\cos\alpha\\&+\varepsilon^2[(3\cos^2\alpha/2)\psi^2-\cos^2\alpha]\\&+\varepsilon^3[\text{function of }(\varphi,\psi)]+\dots\end{aligned} \tag{4.20}$$

Carrying out (ii) we find

$$\begin{aligned}Q_{\text{outer}}\to\varphi^*\cos\alpha&-(3\cos^2\alpha/2)\varphi^{*2}+O(\varphi^{*3})\\&+\varepsilon[c_1+\varphi^*(\sin\alpha\cos\alpha-3c_1\cos\alpha)+O(\varphi^{*2})]\\&+\varepsilon^2[(3\cos^2\alpha/2)\psi^2-(8\cos^2\alpha-6c_1\sin\alpha+9c_1^2)/6\\&\quad+c_2+O(\varphi^*)]+O(\varepsilon^3)\,.\end{aligned} \tag{4.21}$$

On comparing these expressions, matching is achieved if we choose

$$c_1=0 \quad \text{and} \quad c_2=\cos^2\alpha/3\,. \tag{4.22}$$

The terms of $O(\varepsilon^3)$ in (4.20), when written in terms of ε, φ^*, ψ, will not contribute to any of the terms already appearing, although additional terms of the form φ^{*3}, $\varepsilon\varphi^{*2}$, $\varepsilon^2\varphi^*$, etc., will arise.

Composite expansions for Q and θ

Using the inner and outer expansions we may construct composite expansions for Q and θ which are uniformly valid to $O(\varepsilon^2)$ [see Ref. 5, p. 94]. We find

$$Q \sim Q_0(\varphi^*) + \varepsilon Q_1(\varphi^*) + \varepsilon^2 [Q_2(\varphi^*, \psi) + Q^+(\varphi, \psi) - (3\cos^2\alpha/2)(\psi^2 - \varphi^2) - \varphi \sin\alpha \cos\alpha + \cos^2\alpha] + o(\varepsilon^2) \tag{4.23}$$

$$\theta \sim \varepsilon\theta_0(\varphi^*, \psi) + \varepsilon^2 [\theta_1(\varphi^*, \psi) + \theta^+(\varphi, \psi) - 3\varphi\psi \cos^2\alpha + \psi \sin\alpha \cos\alpha] + o(\varepsilon^2). \tag{4.24}$$

These will be used in the next section to deduce various properties of the flow.

5. Discussion of the results

It should be noted that the volumetric flux m is not exactly equal to $b_0 q_0$ because the real and imaginary parts of the analytic function $\Gamma = Q - i\theta$ [see (2.2)] cannot both be specified along the line $X=0$. The exact volumetric flux crossing the line $X=0$ is given by

$$m = \int_0^{b_0} U(0, Y)dY = \int_0^{b_0} q(0, Y)\cos\theta(0, Y)dY = b_0 q_0 + O(\varepsilon^2), \tag{5.1}$$

where we have used (4.1), (4.2), and (4.6).

The non-dimensional streamline curvature is given by

$$b_0/R = e^Q \partial\theta/\partial\varphi \approx \varepsilon^2 e^{Q_0}[\partial\theta_0/\partial\varphi^* + \partial\theta^+/\partial\varphi - 3\psi\cos^2\alpha] + o(\varepsilon^2) = O(\varepsilon^2) \text{ uniformly.} \tag{5.2}$$

When $\varphi^* \to \infty$, the coefficient of ε^2 tends to zero.

The pressure derivatives may be found from (2.7) using the chain rule. We find

$$(\rho q_0^2)^{-1}(\partial p/\partial x) = \varepsilon\cos\alpha - e^{3Q}[\cos\theta(\partial Q/\partial\varphi) - \sin\theta(\partial Q/\partial\psi)], \tag{5.3}$$

$$(\rho q_0^2)^{-1}(\partial p/\partial y) = -\varepsilon\sin\alpha - e^{3Q}[\sin\theta(\partial Q/\partial\varphi) - \cos\theta(\partial Q/\partial\psi)]. \tag{5.4}$$

Substituting the composite expansions (4.23) and (4.24) and using (3.15), (3.18), and (3.19), we obtain

$$(\rho q_0^2)^{-1}(\partial p/\partial x) = -\varepsilon^2 e^{3Q_0}[Q_0' e^{-3Q_0}\sin\alpha + (\partial Q^+/\partial\varphi) + 3\varphi\cos^2\alpha - \sin\alpha\cos\alpha] + O(\varepsilon^3), \tag{5.5}$$

$$(\rho q_0^2)^{-1}(\partial p/\partial y) = -\varepsilon\sin\alpha - \varepsilon^2 e^{3Q_0}[-\psi Q_0'^2 + (\partial Q_2/\partial\psi) + (\partial Q^+/\partial\psi) - 3\psi\cos^2\alpha] + O(\varepsilon^3). \tag{5.6}$$

Except for the hydrostatic term of $O(\varepsilon)$ in (5.6), the pressure derivatives are of $O(\varepsilon^2)$ uniformly.

To summarize we have shown:

1. The hydraulic approximation allows Q and θ to be determined uniformly to $O(\varepsilon)$ by a straightforward iterative procedure provided the initial condition is used to determine an arbitrary constant in the second approximation. The third approximation does not satisfy the initial condition and an inner solution, valid near the leading edge, is required.
2. The streamline curvature, non-dimensionalized with respect to the initial film thickness, is of $O(\varepsilon^2)$ uniformly and depends to that order on the second order inner solution.
3. The pressure derivatives, aside from a hydrostatic term of $O(\varepsilon)$, are of $O(\varepsilon^2)$ uniformly.

6. Acknowledgements

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